§ 1. Convolution

M, N : oriented manifolds (compact for a moment, but noncompact later)

K : differential form on $M \times N$ gives an operator

$$\{\text{diff. forms on } N\} \longrightarrow \{\text{diff. forms on } M\}$$

$$\alpha \longmapsto K * \alpha = \int_{y \in N} K(x, y) \wedge \alpha(y)$$

If K is closed, it descends to

$$\begin{array}{ccc} H^*(N) & \longrightarrow & H^*(M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\alpha] & & [K * \alpha] \end{array}$$

and it depends only on the cohomology class $[K]$ of K.

composition $$\underbrace{M_1 \times M_2}_{K_{12}} \times M_3 \quad M_1 \times \underbrace{M_2 \times M_3}_{K_{23}}$$

$$K_{12} * (K_{23} * \alpha_3) = \int_{M_2} K_{12}(x_1, x_2) \int_{M_3} K_{23}(x_2, x_3)\, \alpha_3(x_3)$$

$$= \int_{M_3} \left\{ \underbrace{\int_{M_2} K_{12}(x_1, x_2) K_{23}(x_2, x_3)}_{\text{differential form on } M_1 \times M_3} \right\} \alpha_3(x_3)$$

$$K_{12} * K_{23} := \int_{M_2} K_{12}(x_1, x_2) K_{23}(x_2, x_3) \quad \text{convolution product}$$

Take $M_1 = M_2 = M_3 = M$

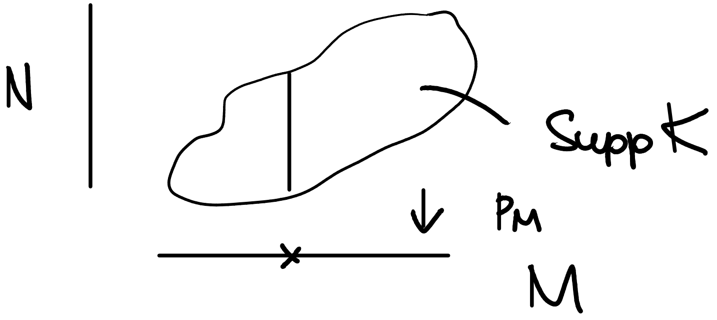
$\Rightarrow$ $H^*(M \times M)$ is an algebra and $H^*(M)$ is its representation.

But it is <u>not</u> interesting, as $H^*(M \times M) \cong H^*(M) \otimes H^*(M) \cong \mathrm{End}\, H^*(M)$

$\therefore$ $H^*(M \times M)$: matrix algebra , $H^*(M)$: vector representation

Suppose M, N : noncompact

$\int_N K(x,y) \wedge \alpha(y)$ does not converge in general

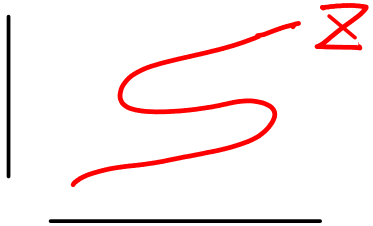


<u>Condition</u>

- $\operatorname{Supp} K \cap$ fiber of p_M is compact.

$\Longrightarrow \int_N K(x,y) \wedge \alpha(y)$ converges

For (co)homology



$Z \subset M \times N$ closed submanifold

<u>condition</u> $Z \xrightarrow[p_M]{} M$ is proper

$\Longrightarrow H^*(N) \xrightarrow{p_N^*} H^*(S) \xrightarrow{p_{M*}} H^*(M)$

$\uparrow$

well-defined

e.g. $M = N$ $Z = \Delta \subset M \times M$ diagonal

$[Z]$ gives an identity operator on $H^*(M)$

<u>Rem</u> Poincaré duality for (oriented) noncompact manifold

$$H_c^*(M) \cong H_{\dim_{\mathbb{R}} M - *}(M) \qquad \alpha \mapsto \alpha \cap [M]$$

$$H^*(M) \cong H^{BM}_{\dim_{\mathbb{R}} M - *}(M)$$

Borel-Moore homology = homology of locally finite chains

$$f: M \to N \quad \text{proper} \quad \Rightarrow \quad H_*^{BM}(M) \to H_*^{BM}(N)$$

$T^*Gr(m,k) \times T^*Gr(m+1,k) \supset$ conormal bundle to the correspondence

$\{(S_1,S_2) \mid S_1 \subset S_2\} \subset Gr(m,k) \times Gr(m+1,k)$

$T^*Gr(m,k) = \{(\xi_1, S_1) \mid \xi_1 : \mathbb{C}^k/S_1 \to S_1\}$

$= \{(S_1,S_2,\xi) \mid \begin{array}{l} S_1 \subset S_2 \\ \xi : \mathbb{C}^k/S_2 \to S_1 \end{array}\}$

In general, we need to consider the case when Z has singularities
e.g. M, N : cpx mfd, Z : analytic subvariety in $M \times N$

Poincaré duality : $H_*^{BM}(S) = H^{\dim M - *}(M \times N, M \times N \setminus Z)$ fund. class $[Z]$ is defined

Remark symplectic geometry

M, N : symp. mfd $\supset Z$ lagrangian

quantization

$\rightsquigarrow$ $\mathcal{H}(M\times N) = \mathcal{H}(M)\otimes\mathcal{H}(N)$ Hilbert space $\ni \mathcal{H}(Z)$

$\simeq \mathrm{Hom}(\mathcal{H}(M), \mathcal{H}(N))$ operator

More concretely, symp / lagr. assumption gives up a natural degree convention

$$[Z] \in H_{\frac{1}{2}(\dim_{\mathbb{R}} M + \dim_{\mathbb{R}} N)}(Z)$$

$$\Rightarrow H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} N + *}(N) \xrightarrow{[Z]*} H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(M)$$

check

$$\begin{array}{ccc} H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} N + *}(N) & & \\ \wr\wr & & \\ H^{\frac{1}{2}\dim_{\mathbb{R}} N - *}(N) & \xrightarrow{p_2^*} & H^{\frac{1}{2}\dim_{\mathbb{R}} N - *}(Z) \\ & & \wr\wr \\ & & H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(Z) \xrightarrow{p_{1*}} H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(M) \end{array}$$

$$M \xleftarrow{p_1} Z \xrightarrow{p_2} N$$

§3. Hilbert scheme of points on cpx surfaces

X : a complex surface (nonsingular)

$n \in \mathbb{Z}_{\geq 0}$ $\quad X^{[n]}$: Hilbert scheme of n points on X

e.g. $X = \mathbb{C}^2$ $\quad \mathbb{C}[x,y] =$ polynomial functions on X

$p_1, \cdots, p_n$: distinct n points

$\rightsquigarrow$ $I = \{ f \in \mathbb{C}[x,y] \mid f(p_i) = 0 \quad i = 1, \cdots, n \}$

ideal in $\mathbb{C}[x,y]$ $\quad \mathbb{C}[x,y]/I$ has $\dim = n$

$$(\mathbb{C}^2)^{[n]} = \{ I \subset \mathbb{C}[x,y] \mid \text{ideal, colength} = n \}$$

$n = 2$ $\quad I = \{ f(0) = 0 , \ \partial f/\partial x (0) = 0 \} = \langle x^2, y \rangle$

$\quad \cancel{\wedge}$

$\quad I' = \{ f(0) = 0 , \ \partial f/\partial y (0) = 0 \} = \langle x, y^2 \rangle$

$$(\mathbb{C}^2)^{[2]} = \{ \text{distinct 2 pts} \} \amalg \mathbb{P}^1 \times \mathbb{C}^2$$

<u>Fact</u> (from the construction) $\quad \pi : X^{[n]} \longrightarrow S^n X$

(Fogarty) $\quad X^{[n]}$: smooth

(Beauville) $\quad X$: symplectic $\Rightarrow X^{[n]}$: symplectic

Th. Assume X cpt for simplicity.

$\bigoplus_{n=0}^{\infty} H_*(X^{[n]})$ is a representation of ∞^d Heis. modelled over $H_*(X)$

$\alpha \in H_*(X) \rightsquigarrow P_k(\alpha)$ $\quad k \in \mathbb{Z}\setminus 0$ $\quad$ $k>0$ annihilation, $<$ creation, s.t.

$$[P_k(\alpha), P_l(\beta)] = (-1)^{k-1} k\, \delta_{k+l,0} (\alpha, \beta)\, \mathrm{id}$$

construction $\quad X \times X^{[n]} \times X^{[n+k]} \supset Z = \{(x, Z_1, Z_2) \mid \pi(Z_2) = \pi(Z_1) + k\cdot x\}$

Fact $\quad Z : \frac{1}{2}\dim_{\mathbb{C}} (= 2n+k+1)$ in $X \times X^{[n]} \times X^{[n+k]}$

singular in general

e.g. $\quad n=0,\ k=1 \qquad X \times \underset{\substack{\| \\ \mathrm{pt}}}{X^{[0]}} \times X \supset$ diagonal Δ_X

$n=0,\ k=2 \qquad X \times X^{[0]} \times \underset{\substack{\cup \\ X\times\mathbb{P}^1}}{X^{[2]}} \supset \Delta_X \times \mathbb{P}^1$

Now the construction is same as symmetric power case.

$$\begin{tikzcd} X^{[n]} \times X^{[n+k]} \times X \arrow[r, "p_X"] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & X \\ X^{[n]} & X^{[n+k]} \end{tikzcd}$$

$$P_k(\alpha) = p_{1*}(p_2^*(\cdot) \cap p_X^*(\alpha) \cap [Z])$$

$$P_{-k}(\alpha) = p_{2*}(p_1^*(\cdot) \cap p_X^*(\alpha) \cap [Z])$$

* $\cap$ is taken in $X^{[n]} \times X^{[n+k]} \times X$

Then we check the relation.

- $k+l \neq 0$ case is easy
- $k+l=0$ $k(-1)^{k-1}$ is delicate (cf. $k=2$ $\mathbb{P}^1 \subset T^*\mathbb{P}^1$, $[\mathbb{P}^1]^2 = -2$

  but can be done.

$\Gamma \subset SL(2,\mathbb{C})$ finite subgrp ADE classification

- cyclic
- binary dihedral
- binary polyhedral

$\mathbb{C}^2 \curvearrowleft \Gamma$ induces $(\mathbb{C}^2)^{[n]} \curvearrowleft \Gamma$

<u>Th.</u> $\bigoplus_{n=0}^{\infty} H_{mid}\left(((\mathbb{C}^2)^{[n]})^{\Gamma}\text{-fixed point} \right)$ has a structure of the basic representation of an affine Lie algebra of $\mathfrak{g}_{ADE}$

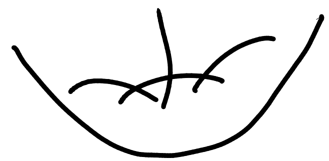
(highest weight $= \Lambda_0$

e.g. $n = \#\Gamma$ $((\mathbb{C}^2)^{[\#\Gamma]})^{\Gamma} \supset \Gamma$-orbit of a point in $\mathbb{C}^2 \setminus \{0\}$

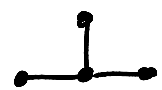
$$\Gamma\text{-orbit of a point in } \mathbb{C}^2\setminus\{0\} \;=\; (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

There is a component $M(\delta) \subset ((\mathbb{C}^2)^{[\#\Gamma]})^{\Gamma}$

$$M(\delta) \xrightarrow{\text{resolution of singularities}} \mathbb{C}^2/\Gamma$$



Dynkin diagram



$H_{mid}(M(\delta)) \cong \mathfrak{h}$: Cartan subalg. of $\mathfrak{g}$ $\subset$ basic rep.

Construction

α : representation of Γ

$$M(\alpha) = \{ I \subset \mathbb{C}[x,y] \mid \Gamma\text{-invariant ideal},\ \mathbb{C}[x,y]/I \cong \alpha \text{ as rep. of } \Gamma \}$$

(δ = regular rep. of Γ)

ρ_i : irreducible rep. of Γ

McKay $\rho_i \longleftrightarrow$ vertex of affine Dynkin diagram

$\longleftrightarrow e_i, f_i$: generators of affine Lie alg.

$$Z \subset M(\alpha) \times M(\alpha+\rho_i)$$

$$= \{ (I_1, I_2) \mid I_1 \supset I_2 \}$$

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M(\alpha) & & M(\alpha+\rho_i) \end{tikzcd}$$

$$f_i = p_{2*}(p_1^*(\ \ ))$$

$$e_i = p_{1*}(p_2^*(\ \ ))$$

§4. equivariant (co)homology

4.1 X : variety with $\mathbb{G}_m$ (more generally T) action

$\Longrightarrow H^*_{\mathbb{G}_m}(X)$, $H^{\mathbb{G}_m}_*(X)$ ($\mathbb{C}$-coefficients) satisfying funcorial properties as usual (co)homology groups

Take $V = \mathbb{C}^N \curvearrowleft \mathbb{G}_m$ $(N \gg 0)$

Note · $V \setminus \{0\} \curvearrowleft \mathbb{G}_m$ free

· $H^i(V \setminus \{0\}) = H^i(S^{2N-1}) = 0$ except $i = 0, 2N-1$

Consider $X_V := X \times (V \setminus \{0\}) / \mathbb{G}_m \xleftarrow[\text{principal } \mathbb{G}_m\text{-b'dle}]{} X \times (V \setminus \{0\})$, and

set $H^i_{\mathbb{G}_m}(X) := H^i(X_V)$.

○ independence of V if $\dim V \gg 0$

∵ $X_{V_1} \longleftarrow X \times (V_1 \setminus \{0\}) \times (V_2 \setminus \{0\}) / \mathbb{G}_m \longrightarrow X_{V_2}$

fiber bundles with fibers $V_2 \setminus \{0\}$, $V_1 \setminus \{0\}$ respectively.

no cohomology except 0 & very large

$\Rightarrow H^i(X_{V_1}) \cong H^i(\text{middle}) \cong H^i(X_{V_2})$ //

○ $f : X \to Y$ $\mathbb{G}_m$-equivariant morphism
$\Longrightarrow$ $f^* : H^*_{\mathbb{G}_m}(Y) \longrightarrow H^*_{\mathbb{G}_m}(X)$

○ Suppose $X \curvearrowleft \mathbb{G}_m$ free. Then $X_V \longrightarrow X/\mathbb{G}_m$ is a fiber b'dle with fiber $V \setminus \{0\}$
no cohomology except $0, 2N-1$.

$\Longrightarrow$ $H^i_{\mathbb{G}_m}(X) \cong H^i(X/\mathbb{G}_m)$

○ E : $\mathbb{G}_m$-equivariant vector bundle $\Longrightarrow$ $c_i(E) \in H^{2i}_{\mathbb{G}_m}(X)$ equivariant Chern class

○ $H^*_{\mathbb{G}_m}(X)$ has the cup product $H^i_{\mathbb{G}_m}(X) \otimes H^j_{\mathbb{G}_m}(X) \longrightarrow H^{i+j}_{\mathbb{G}_m}(X)$
$f^* : H^*_{\mathbb{G}_m}(Y) \longrightarrow H^*_{\mathbb{G}_m}(X)$ is a ring hom.

Ex. $X = pt$ L : canonical 1 dim rep. of $\mathbb{G}_m$ $\longrightarrow$ equivariant vector b'dle over X

$X_V = \mathbb{P}(V)$: $L_V = \mathcal{O}(1)$ over X_V
Let $\hbar := c_1(L_V) \in H^2(X_V)$
Then $H^*(X_V) \cong \mathbb{C}[\hbar]/(\hbar^N = 0) \overset{N\to\infty}{\rightsquigarrow} H^*_{\mathbb{G}_m}(pt) \cong \mathbb{C}[\hbar]$.

- $X \longrightarrow pt \quad \Rightarrow \quad H^*_{\mathbb{G}_m}(pt) \longrightarrow H^*_{\mathbb{G}_m}(X)$

  $\therefore \; H^*_{\mathbb{G}_m}(X)$ is a module over $H^*_{\mathbb{G}_m}(pt) = \mathbb{C}[\hbar]$

- $X_V = X \times (V \setminus \{0\}) / \mathbb{G}_m \quad \Longrightarrow \quad H^*_{\mathbb{G}_m}(X) \longrightarrow H^*(X)$

  $\downarrow$ fiber X

  $\mathbb{P}(V)$

  restriction to the fiber (forgetting hom)

<u>Rem</u> torus case : $T = \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{\ell} \Rightarrow$ Use $V = \mathbb{C}^{N\ell}$: T-module

$$H^*_T(pt) = \mathbb{C}[\hbar_1, \cdots, \hbar_\ell] = \mathbb{C}[\mathrm{Lie}\,T] \; (\text{polynomial on } \mathrm{Lie}\,T)$$

$T' \subset T$ subtorus $\Longrightarrow H^*_T(X) \longrightarrow H^*_{T'}(X)$ restriction hom

Given by pull-back

$$X \times (V \setminus \{0\})/T \longleftarrow X \times (V' \setminus \{0\})/T'$$

In particular, we have
$$\begin{array}{ccc} H^*_T(pt) & \longrightarrow & H^*_{T'}(pt) \\ \| & & \| \\ \mathbb{C}[\mathrm{Lie}\,T] & \longrightarrow & \mathbb{C}[\mathrm{Lie}\,T'] \end{array}$$

This is induced by $\mathrm{Lie}\,T' \longrightarrow \mathrm{Lie}\,T$.

4.3 localization thm

Consider $H_T^*(X), H^T_*(X)$ as modules over $H_T^*(pt) = \mathbb{C}[\mathrm{Lie}\,T]$

Suppose $\mathrm{Stab}\,x = T'$ $\forall x \in X$ $\quad X \curvearrowleft T/T'$ free

<u>Claim</u> $\mathrm{Supp}\, H_T^*(X) \subset \mathrm{Lie}\,T' \quad (\subset \mathrm{Lie}\,T)$

(∵) $X_V = X \times V\setminus\{0\}/T = X \times (V\setminus\{0\}/T')/T/T' \xrightarrow[\text{fiber} = V\setminus\{0\}/T']{} X/T/T'$

∴ locally $X_V = V\setminus\{0\}/T' \times X/T/T'$

∴ $X_V \longrightarrow V\setminus\{0\}/T$ factors through $V\setminus\{0\}/T'$

∴
$$\begin{tikzcd}
H^*(V\setminus\{0\}/T) \arrow[rr] \arrow[d, "V\to\infty"'] \arrow[dr] & & H^*(X_V) \\
\mathbb{C}[\mathrm{Lie}\,T] & H^*(V\setminus\{0\}/T') \arrow[ur] \arrow[dl] & \\
\mathbb{C}[\mathrm{Lie}\,T'] & &
\end{tikzcd}$$

Generally $X \curvearrowleft T$ can be stratified by Stab

$\mathrm{Stab}\,x = T \iff x$: fixed pt

∴ $X \setminus X^T$ has stab $\subsetneq T \implies \mathrm{Supp}\, H_T^*(X \setminus \overline{X^T}) \subsetneq \mathrm{Lie}\,\mathbb{C}$

Th. (localization) $H_T^*(X) \xrightarrow{i^*} H_T^*(X^T) \cong H_T^*(pt) \otimes H^*(X^T)$

is an isomorphism over the generic point of Lie T,

i.e. $\otimes_{\mathbb{C}[\mathrm{Lie}\,T]} \mathbb{C}(\mathrm{Lie}\,T)$

quotient field

The same is true for $H_*^T(X^T) \xrightarrow{i_*} H_*^T(X)$

4.4 Fixed point formula

Assume X : nonsingular $\therefore$ Poincaré duality $H_T^*(X) \cong H^T_{2\dim X - *}(X)$

X^T : also nonsingular $= \coprod X_\alpha$: connected component

$i^* i_* : H_*^T(X^T) \longrightarrow H_*^T(X^T)$ preserves $H_*^T(X_\alpha)$.

$H_*^T(X^T) \cong H_T^*(pt) \otimes H_*(X^T)$

Let $i_\alpha = i|_{X_\alpha}$

<u>Prop</u> $i_\alpha^* i_{\alpha *}\Big|_{H_*^T(X_\alpha)} = e(N_\alpha) \frown \cdot$

N_α = normal bdle, $e(N_\alpha)$ = equivariant Euler (top Chern) class

<u>Lemma</u>. $e(N_\alpha)$ is invertible in $H_*^T(X_\alpha) \otimes_{H_*^T(pt)} \mathbb{C}(\mathrm{Lie}\,T)$

$\because$ $H_T^*(X_\alpha) = H_T^*(pt) \otimes H^*(X_\alpha)$

$H_T^*(pt) \otimes H^{>0}(X_\alpha)$ nilpotent as $H^{>2\dim X_\alpha}(X_\alpha) = 0$.

$\therefore$ enough to check that the $H_T^*(pt) \otimes H^0(X_\alpha) \cong \mathbb{C}[\mathrm{Lie}\,T]$ component of $e(N_\alpha) \neq 0$

Take $x \in X_\alpha$ $\quad T_x X \curvearrowleft T$-module $= \bigoplus_\lambda V_\lambda$

$\lambda : T \to \mathbb{C}^*$ weight $\quad V_\lambda = \{ v \in T_x X \mid t \cdot v = \lambda(t) v \}$

Then $T_x X_\alpha = V_1$, $\quad N_\alpha = \bigoplus_{\lambda \neq 1} V_\lambda$

$\mathbb{C}[\mathrm{Lie}\,T]$-component of $e(N_\alpha) = \prod_{\lambda \neq 1} d\lambda^{\dim V_\lambda}$

where $d\lambda : \mathrm{Lie}\,T \longrightarrow \mathbb{C} \quad \in \mathbb{C}[\mathrm{Lie}\,T]$

Since $\lambda \neq 1 \Rightarrow d\lambda \neq 0$, therefore $\prod \neq 0$ //

<u>Th</u> Assume X : nonsingular and proper $\quad a : X \longrightarrow pt$

$\omega \in H_*^T(X)$

$$\mathbb{C}[\mathrm{Lie}\,T] \ni \int_X \omega = a_* \omega = \sum_\alpha \int_{X_\alpha} e(N_\alpha)^{-1} i_\alpha^* \omega \in \mathbb{C}(\mathrm{Lie}\,T)$$

$$a_* \omega = a_* \sum_\alpha i_{\alpha *} i_{\alpha *}^{-1} \omega = \sum_\alpha \underbrace{(a \circ i_\alpha)_*}_{= \int_{X_\alpha}} i_\alpha^{-1} \omega$$

$$i_\alpha^* \omega = i_\alpha^* i_{\alpha *} i_{\alpha *}^{-1} \omega = e(N\alpha) \cap i_{\alpha *}^{-1} \omega$$

$$\therefore \quad i_{\alpha *}^{-1} \omega = e(N\alpha)^{-1} i_\alpha^* \omega$$

//